Outline

Goal: Noether's (1^{st}) Thm

- What is it?
- Principle of Least Action & Lagrangian
  - ↳ Calculus of Variation
- Derive Euler Lagrange (EL) Equation
- EL ⇒ Newton's 2^{nd} law
- "Physicist Proof" of Noether Thm
  - ↳ Examples

Noether's 1^{st} Law

Symmetry $\Longrightarrow$ Conservation Law

$\hookrightarrow$ energy conservation

Classical Mechanics

Slogan: "Nature is lazy!"

Def: Action

find stationary action

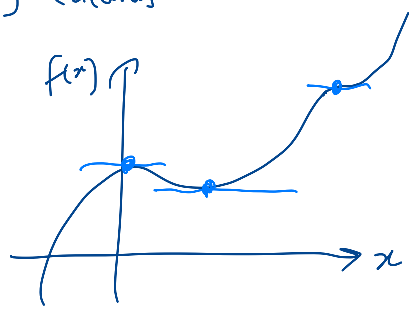
$$\boxed{S=\int_{t_1}^{t_2}\mathcal{L}(x,\dot{x})\,dt}$$

Lagrangian

position

velocity

1D calculus



side note:
For simple harmonic oscillator,
action is max

kinetic

potential

e.g. $\mathcal{L} = T - V$

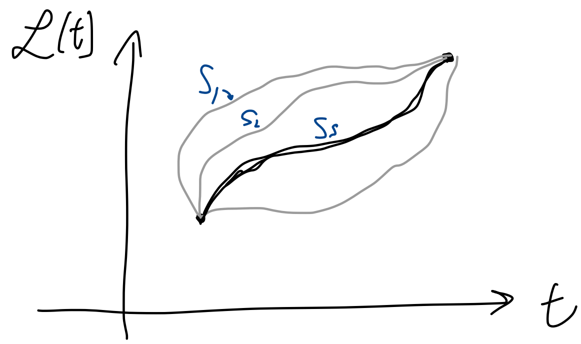
for class. mech.

Calculus of variation

$S[\mathcal{L}]$

↑ functional

aim: vary $\mathcal{L}$ to find stationary pt for S



$$S = \int dt \; \mathcal{L}(x, \dot{x})$$

$$\delta S = \int dt \; \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x}$$

variation

Use physics: $\delta \dot{x} = \frac{d}{dt}(\delta x)$

$$\delta S = \int dt \; \frac{\partial \mathcal{L}}{\partial x} \delta x + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{d}{dt}(\delta x)}$$

Integration by part: $\underbrace{\cancel{\left.\frac{\partial \mathcal{L}}{\partial \dot{x}} \delta x \right|_{\text{boundary}}} \to 0 \quad - \int dt \; \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \delta x}$

$$\underset{\substack{\parallel \\ 0}}{\delta S} = \int dt \; \delta x \left[\underbrace{\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)}}_{\substack{\parallel \\ 0}} \right]$$

EL Eq!

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

NTS EL eq. $\Rightarrow$ N2 law

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$\boxed{F = ma\,!}$$

$$LHS = -\frac{\partial V(x)}{\partial x} = F$$

(conservative law)

$$RHS = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2\right)\right) = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x} = ma$$

"Physicist Proof" of Noether Thm   symmetry $\Rightarrow$ Conservation law

$$\text{symmetry} \Rightarrow \delta \mathcal{L}(x, \dot{x}) = 0$$

$$\delta \mathcal{L}(x,\dot{x}) = \frac{\partial \mathcal{L}}{\partial x}\delta x + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}} \underbrace{\delta \dot{x}}_{\frac{d}{dt}(\delta x)}}_{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta x}$$

$$\delta \mathcal{L} = \delta x \left(\underbrace{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)}_{\text{EL eq.}} \right) + \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\delta x \right)}_{\text{Conservation}}$$

In classical field theory

Spacetime interval $(t, \vec{x})$

$$\Delta s^2 = (\Delta t)^2 - (\Delta \vec{x})^2$$

scalar field $\varphi(\vec{x})$ 4 vector (t, x, y, z)

Aside: index notation:

$$\partial_\mu \varphi(x) = \begin{pmatrix} \frac{\partial \varphi}{\partial t} \\ -\frac{\partial \varphi}{\partial x} \\ -\frac{\partial \varphi}{\partial y} \\ -\frac{\partial \varphi}{\partial z} \end{pmatrix}$$

$$g = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g^{\mu\nu}$$

$$x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_\mu x^\mu = x_\mu (g^{\mu\nu} x_\nu) = g_{00} x_0 x_0 + g_{11} x_1 x_1 + g_{22} x_2 x_2 + g_{33} x_3 x_3$$
$$= (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2$$

$$\mathcal{L}(\varphi, \partial_\mu \varphi)$$

$$\delta \mathcal{L} = \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right]}_{\text{EL for field}} \delta\varphi + \underbrace{\partial_\mu \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial_\mu \varphi} \delta\varphi}_{J^\mu} \Big)}$$

$$J^\mu = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$$

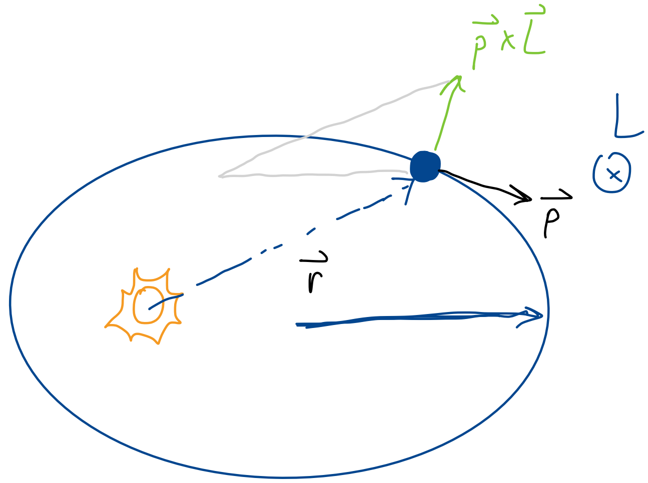
$$\partial_\mu J^\mu = 0$$

electric current

charge conservation $\boxed{\frac{\partial \rho}{\partial t} - \nabla \cdot J = 0}$

Laplace - Runge - Lenz vector

$\frac{1}{r}$ potential



In 4D, a free particle constrained on S^3 ← 3D sphere

Symmetry	Conservation
spatial translation	momentum
spatial rotation	angular momentum
time translation	energy